

Realizability games for the specification problem

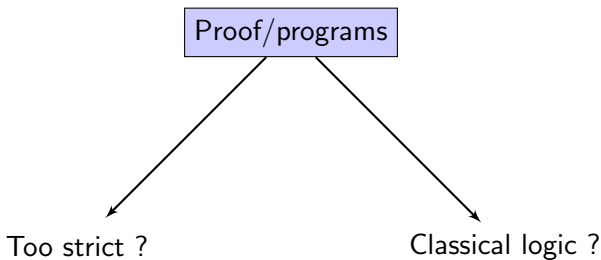
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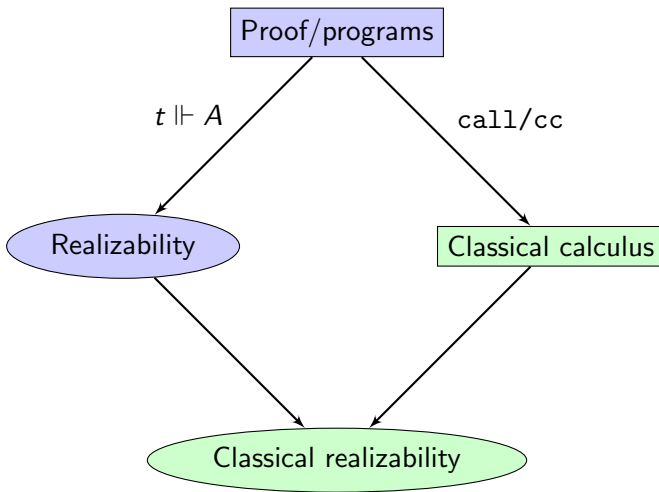
Realizability in Piriápolis



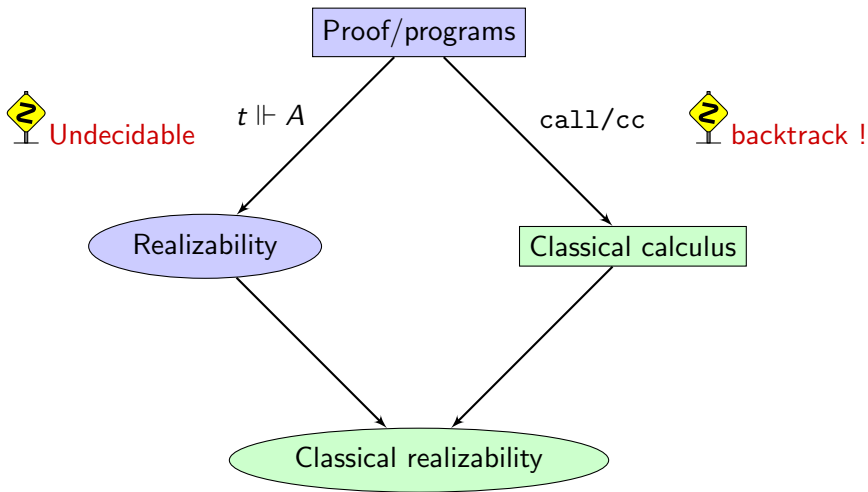
Starting picture



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The question of this talk

Specification of A :

Can we give a **characterization** of the realizers of A ?

Subtitle:

Extract *computational* properties from a *logical* information

(and not the other way round).

Focus of this talk

Arithmetic formulæ:

$$\exists x_1 \forall y_1 \dots \exists x_h \forall y_h f(\vec{x}, \vec{y}) = 0$$

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(A quick reminder about)

Krivine classical realizability

λ_C -calculus

Terms, stacks, processes

\mathcal{B} : stack constants

\mathcal{C} : instructions (including \mathbf{c}), countable

Terms	t, u	$::=$	$x \mid \lambda x.t \mid tu \mid \mathbf{k}_\pi \mid \kappa$	$\kappa \in \mathcal{C}$
Stacks	π	$::=$	$\alpha \mid t \cdot \pi$	$(\alpha \in \mathcal{B}, t \text{ closed})$
Processes	p, q	$::=$	$t \star \pi$	$(t \text{ closed})$

KAM

:

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KAM

PUSH	:	$(t)u \star \pi$	γ_1	$t \star u \cdot \pi$
GRAB	:	$\lambda x. t \star u \cdot \pi$	γ_1	$t\{x := u\} \star \pi$
SAVE	:	$\alpha \star t \cdot \pi$	γ_1	$t \star \mathbf{k}_\pi \cdot \pi$
RESTORE	:	$\mathbf{k}_\pi \star t \cdot \rho$	γ_1	$t \star \pi$

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KAM + \mathcal{C} extended

SAVE :	$\alpha \star t \cdot \pi$	γ_1	$t \star \mathbf{k}_\pi \cdot \pi$
QUOTE :	$\text{quote} \star \phi \cdot t \cdot \pi$	γ_1	$t \star \bar{n}_\phi \cdot \pi$
FORK :	$\uparrow \star t \cdot u \cdot \pi$	γ_1	$t \star \pi$
FORK :	$\uparrow \star t \cdot u \cdot \pi$	γ_1	$u \star \pi$
PRINT :	$\text{print} \star \bar{n} \cdot t \cdot \pi$	γ_1	$t \star \pi$

2nd-order arithmetic

Language

Expressions $e ::= x \mid f(e_1, \dots, e_k)$

Formulæ $A, B ::= X(e_1, \dots, e_k) \mid A \Rightarrow B \mid \forall x A \mid \forall X A$

Shorthands :

$$\perp \equiv \forall Z. Z$$

$$\neg A \equiv A \Rightarrow \perp$$

$$A \wedge B \equiv \forall Z((A \Rightarrow B \Rightarrow Z) \Rightarrow Z)$$

$$A \vee B \equiv \forall Z((A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z)$$

$$A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$$

$$\exists x A(x) \equiv \forall Z(\forall x(A(x) \Rightarrow Z) \Rightarrow Z)$$

$$\exists X A(X) \equiv \forall Z(\forall X(A(X) \Rightarrow Z) \Rightarrow Z)$$

$$e_1 = e_2 \equiv \forall Z(Z(e_1) \Rightarrow Z(e_2))$$

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Typing rules

$$\frac{}{\Gamma \vdash x : A} \quad (x : A) \in \Gamma$$

$$\frac{}{\Gamma \vdash t : \top} \quad FV(t) \subset \text{dom}(\Gamma)$$

$$\frac{\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash t : A}{\Gamma \vdash tu : B}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \Rightarrow B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall x. A} \quad x \notin FV(\Gamma)$$

$$\frac{\Gamma \vdash t : \forall x. A}{\Gamma \vdash t : A\{x := e\}}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall X. A} \quad X \notin FV(\Gamma)$$

$$\frac{\Gamma \vdash t : \forall X. A}{\Gamma \vdash t : A\{X(x_1, \dots, x_k) := B\}}$$

$$\frac{}{\Gamma \vdash \mathbf{c} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}$$

Semantics

Intuition

- falsity value $\|A\|$: **stacks**, **opponent** to A
- truth value $|A|$: **terms**, **player** of A
- pole $\perp\!\!\!\perp$: **processes**, **referee**

$$t \star \pi \succ p_0 \succ \dots \succ p_n \in \perp\!\!\!\perp?$$

$\rightsquigarrow \perp\!\!\!\perp \subset \Lambda_c \star \Pi$ closed by anti-reduction

Truth value defined by **orthogonality** :

$$|A| = \|A\|^{\perp\!\!\!\perp} = \{t \in \Lambda_c : \forall \pi \in \|A\|, t \star \pi \in \perp\!\!\!\perp\}$$

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Models ($\mathcal{M}, \perp\!\!\!\perp$)

Ground model \mathcal{M}

Pole

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$$\forall p, p' \in \Lambda_c \star \Pi : (p \succ p') \wedge (p' \in \perp\!\!\!\perp) \Rightarrow p \in \perp\!\!\!\perp$$

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- $\|A \Rightarrow B\| = \{t \star \pi : t \in |A| \wedge \pi \in \|B\|\}$
- $\|\forall x A\| = \bigcup_{n \in \mathbb{N}} \|A\{x := n\}\|$
- $\|\forall X A\| = \bigcup_{F: \mathbb{N}^k \rightarrow \mathcal{P}(\Pi)} \|A\{X := \dot{F}\}\|$
- $\|\dot{F}(e_1, \dots, e_k)\| = F(\|e_1\|, \dots, \|e_k\|)$

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Notation

$$\begin{array}{ll}
 t \Vdash A & \text{iff} \quad t \in |A| = \|A\|^{\perp\!\!\!\perp} \\
 t \Vdash\!\!\!\Vdash A & \text{iff} \quad t \Vdash A \text{ for all } \perp\!\!\!\perp
 \end{array}$$

Remarks

Case $\perp\!\!\!\perp = \emptyset$ (degenerated model)

- Truth as in the standard model:

$$|A| = \begin{cases} \Lambda & \text{if } \llbracket A \rrbracket = 1 \\ \emptyset & \text{if } \llbracket A \rrbracket = 0 \end{cases}$$

- Realizable \Leftrightarrow True in the standard model

Case $\perp\!\!\!\perp \neq \emptyset$

- $t \star \pi \in \perp\!\!\!\perp \Rightarrow$ forall A , $k_\pi t \Vdash A$
- Restriction to **proof-like**

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Properties

Adequacy

$$\left\{ \begin{array}{l} x_1 : A_1, \dots, x_k : A_k \vdash t : A \\ \forall i \in [1, k] (t_i \Vdash A_i) \end{array} \right. \Rightarrow t[t_1/x_1, \dots, t_k/x_k] \Vdash A$$

Realizing Peano axioms

If $PA2^- \vdash A$, then there is a closed proof-like term t s.t. $t \Vdash A$.

Witness extraction

If $t \Vdash \exists^N x A(x)$ and $A(x)$ is atomic or decidable, then we can build a term u s.t. that $\forall \pi \in \Pi$:

$$t \star u \cdot \pi \succ \text{stop} \star \bar{n} \cdot \pi \quad \wedge \quad A(n) \text{ holds}$$

Relativization

$$\text{Nat}(x) \equiv \forall Z (Z(0) \Rightarrow \forall y (Z(y) \Rightarrow Z(s(y)))) \Rightarrow Z(x)$$

Proposition

There is no $t \in \Lambda_c$ such that $t \Vdash \forall n. \text{Nat}(n)$

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Fix:

$$\forall^{nat}_x A := \forall x (\text{Nat}(x) \Rightarrow A)$$

Obviously, $\lambda x. x \Vdash \forall^{nat}_x \text{Nat}(x)$

Relativization

$$\text{Nat}(x) \equiv \forall Z (Z(0) \Rightarrow \forall y (Z(y) \Rightarrow Z(s(y)))) \Rightarrow Z(x)$$

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Better :

$$\begin{aligned}
 A, B &::= \dots \mid \{e\} \Rightarrow A \\
 \|\{e\} \Rightarrow A\| &= \{\bar{n} \cdot \pi : \llbracket e \rrbracket = n \wedge \pi \in \llbracket A \rrbracket\} \\
 \forall^N x A(x) &\equiv \forall x (\{x\} \Rightarrow A(x))
 \end{aligned}$$

Let T be a storage operator. The following holds for any formula $A(x)$:

- ① $\lambda x. x \Vdash \forall^{nat} x A(x) \Rightarrow \forall^N x A(x)$
- ② $\lambda x. Tx \Vdash \forall^N x A(x) \Rightarrow \forall^{nat} x A(x)$

Our problem

Specification of A

Can we give a characterization of $\{t \in \Lambda_c : t \Vdash A\}$?

Absoluteness

Are arithmetical formulæ absolute for realizability models $(\mathcal{M}, \perp\!\!\!\perp)$?

The specification problem

Our leverage: the pole

Two ways of building poles from any set P of processes.

① goal-oriented :

$$\perp\!\!\!\perp := \{p \in \Lambda_c \star \Pi : \exists p' \in P, p \succ p'\}$$

② thread-oriented :

Thread of p : $th_p = \{p' \in \Lambda_c \star \Pi : p \succ p'\}$

$$\perp\!\!\!\perp := \left(\bigcup_{p \in P} th_p \right)^c \equiv \bigcap_{p \in P} th_p^c$$

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Proof:

Let $p_0 \in \Lambda \star \Pi$ and

$$\perp\!\!\!\perp_0 := \{p \in \Lambda_c \star \Pi : p \succ p_0\}.$$

Let $p_1, p_2 \in \Lambda \star \Pi$ be such that:

$$p_1 \succ p_2 \quad \text{and} \quad p_2 \in \perp\!\!\!\perp_0 \equiv p_2 \succ p_0.$$

Then

$$p_1 \succ p_2 \succ p_0,$$

thus $p_1 \in \perp\!\!\!\perp_0$.



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Let $p_0 \in \Lambda \star \Pi$ and

$$\perp_0 := th_{p_0}^c \equiv \{p \in \Lambda_c \star \Pi : p_0 \not\succeq p\}.$$

Let $p_1, p_2 \in \Lambda \star \Pi$ be such that:

$$p_1 \succ p_2 \quad \text{and} \quad p_2 \in \perp_0 \equiv p_0 \not\succeq p_2.$$

Assume $p_0 \succ p_1$, then

$$p_0 \succ p_1 \succ p_2 \notin \perp_0$$

Absurd !✓

Our leverage: the pole

② thread-oriented :

Thread of p : $th_p = \{p' \in \Lambda_c \star \Pi : p \succ p'\}$

$$\perp := \left(\bigcup_{p \in P} th_p \right)^c \equiv \bigcap_{p \in P} th_p^c$$

Proof:

Let $p_0 \in \Lambda \star \Pi$ and

$$\perp_0 := th_{p_0}^c \equiv \{p \in \Lambda_c \star \Pi : p_0 \not\succeq p\}.$$

Let $p_1, p_2 \in \Lambda \star \Pi$ be such that:

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Example: Identity-like

$$t \Vdash \forall X.(X \Rightarrow X) \quad \text{iff} \quad ?$$

By definition:

$$\|\forall X.(X \Rightarrow X)\| = \bigcup_{S \in \mathcal{P}(\mathbb{N})} \{u \cdot \pi : u \Vdash \dot{S} \wedge \pi \in S\}$$

Proof:

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Proof:

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Proof:

(\Leftarrow) Obvious.

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Proof:

(\Rightarrow) Assume $t \Vdash \forall X(X \Leftarrow X)$ and fix $u \in \Lambda$ and $\pi \in \Pi$.

① goal-oriented :

$$\perp\!\!\!\perp := \{p : p \succ u \star \pi\}$$

Amounts to: $u \cdot \pi \in \|\forall X.(X \Rightarrow X)\|$?

Define $S := \{\pi\}$. We check that:

- $\pi \in S$
- $u \Vdash \dot{S} \Leftrightarrow \forall \rho \in S, u \star \rho \in \perp\!\!\!\perp \Leftrightarrow u \star \pi \in \perp\!\!\!\perp$

Thus $t \star u \cdot \pi \in \perp\!\!\!\perp$ and $t \star u \cdot \pi \succ u \star \pi$. ✓

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$$\perp := th_{t \star u \cdot \pi}^c \equiv \{p \in \Lambda_c \star \Pi : t \star u \cdot \pi \not\prec p\}.$$

Obviously, $t \star u \cdot \pi \notin \perp$. Thus

$$u \cdot \pi \notin \|\forall X.(X \Rightarrow X)\|.$$

Defining $S := \{\pi\}$, we deduce that :

$$u \not\prec \dot{S} \Leftrightarrow \exists \rho \in S, u \star \rho \notin \perp \Leftrightarrow u \star \pi \notin \perp \Leftrightarrow t \star u \cdot \pi \succ u \star \pi \quad \checkmark$$

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$t_0 \Vdash \forall X.(X \Rightarrow X) \Rightarrow X \Rightarrow X$ iff ???

$$\begin{array}{l}
 t_0 \star \kappa_S \cdot \kappa_Z \cdot \pi \succ \kappa_S \star t_1 \cdot \pi \\
 t_1 \star \pi \succ \kappa_S \star t_2 \cdot \pi \\
 \vdots \\
 t_i \star \pi \succ \kappa_S \star t_{i+1} \cdot \pi \\
 \vdots \\
 t_S \star \pi \succ \kappa_Z \star \pi
 \end{array}$$

$t_0 \Vdash \forall X.(X \Rightarrow X) \Rightarrow X \Rightarrow X$ iff ???

$$\begin{array}{rcl}
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 & \dots & \\
 t_j \star \pi & \succ & \kappa_S \star t_{j+1} \cdot \pi \\
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- ① Define $p_0 := t_0 \star \kappa_S \cdot \kappa_Z \cdot \pi$, $\perp_0 := (th(p_0))^c$ and $\|X\| = \{\pi\}$:
- $\nrightarrow \kappa_Z \Vdash_0 X$ implies $p_0 \succ \kappa_Z \star \pi$
 - $\nrightarrow \kappa_Z \Vdash_0 X$ implies $\kappa_S \Vdash_0 X \Rightarrow X$ and $p_0 \succ \kappa_S \star t_1 \cdot \pi$

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- $\curvearrowright \kappa_Z \not\Vdash_0 X$ implies $p_0 \succ \kappa_Z \star \pi$
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1 Define $p_i := t_i \star \pi$, $\perp_i := \bigcap_{j \in [0, i]} (th(p_j))^c$ and $\|X\| = \{\pi\}$:

$\curvearrowright \kappa_Z \not\Vdash_i X$ implies $p_i \succ \kappa_Z \star \pi$

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Termination:

If $\forall i \in \mathbb{N} (\kappa_Z \not\Vdash_i X)$, define $\perp_\infty := \bigcap_{i \in \mathbb{N}} (th(p_i))^c$, get a contradiction.

$t_0 \Vdash \forall XY.(X \Rightarrow Y) \Rightarrow X \Rightarrow Y$ iff ???

$$\begin{array}{lcl}
 t_0 * \kappa_f \cdot \kappa_x \cdot \pi & \succ & \kappa_f * t_1 \cdot \pi \\
 t_1 * \pi' & \succ & \kappa_f * t_2 \cdot \pi \\
 & \vdots & \\
 t_i * \pi' & \succ & \kappa_f * t_{i+1} \cdot \pi \\
 & \vdots & \\
 t_s * \pi' & \succ & \kappa_x * \pi'
 \end{array}$$

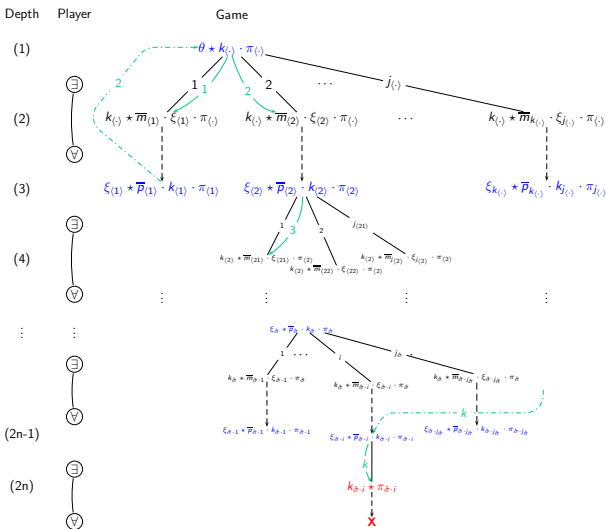
(Same proof)

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(Same proof)

Arithmetical formulæ by hand



Advertisement

Problem

You want to specify A .

Methodology:

\leadsto requirement: some intuition...

- 1 **direct-style**: define the good poles,
- 2 **indirect-style**: try the thread method,
- 3 **induction-style**: define a game

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A first notion of game

Coquand's game

Arithmetical formula

$$\Phi_{2h} : \exists x_1 \forall y_1 \dots \exists x_h \forall y_h f(\vec{x}_h, \vec{y}_h) = 0$$

Rules:

- **Players** : Eloise (\exists) and Abelard (\forall).
- **Moves** : - at his turn, each player instantiates his variable
- **Eloise allowed to backtrack**
- **Final position** : evaluation of $f(\vec{m}_h, \vec{n}_h) = 0$:
 - true : Eloise wins
 - false : game continues
- Abelard wins if the game never ends

Winning strategy

Way of playing that ensures the victory, independently of the opponent moves.

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Way of playing that ensures the victory, independently of the opponent moves.

The need of backtrack

$$f(m, n, p) = 0 \quad := \quad (n > 0 \wedge \text{Halt}(m, n)) \vee (n = 0 \wedge \neg \text{Halt}(m, p))$$

$$\text{Halt}(m, n) \quad := \quad \text{the } m^{\text{th}} \text{ Turing machine stops before } n \text{ steps}$$

Formula

$$\forall m \exists n \forall p (f(m, n, p) = 0)$$

Winning strategy ?

- Abelard plays the code m of a Turing machine \mathcal{M} .
- Eloise chooses to play $n = 0$ (" \mathcal{M} never stops")
- Abelard answers a given number of steps p
- Eloise checks whether \mathcal{M} stops before p steps:
 - either \mathcal{M} is still running after p steps :
 \leadsto Eloise wins.
 - either \mathcal{M} stops before p steps :
 \leadsto Eloise backtracks and plays p instead of 0 (" \mathcal{M} stops before p steps")

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Example

Formula

$$\exists x \forall y \exists z (x \cdot y = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$

Example

Formula

$$\forall y \exists z (1 \cdot y = 2 \cdot z)$$

Player	Action	Position
⊖	Start	$P_0 = (\cdot, \cdot, \cdot)$
	$x := 1$	$P_1 = (1, \cdot, \cdot)$

Example

Formula

$$\exists z(1 = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$

Example

Formula

$$\forall y \exists z (2 \cdot y = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$
\exists	backtrack to $P_0 + x := 2$	$P_3 = (2, \cdot, \cdot)$

Example

Formula

$$\exists z(2 = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$
\exists	backtrack to $P_0 + x := 2$	$P_3 = (2, \cdot, \cdot)$
\forall	$y := 1$	$P_4 = (2, 1, \cdot)$

Example

Formula

$$2 = 2$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$
\exists	backtrack to $P_0 + x := 2$	$P_3 = (2, \cdot, \cdot)$
\forall	$y := 1$	$P_4 = (2, 1, \cdot)$
\exists	$z := 1$	$P_5 = (2, 1, 1)$

Example

Formula

$$2 = 2$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$
\exists	backtrack to $P_0 + x := 2$	$P_3 = (2, \cdot, \cdot)$
\forall	$y := 1$	$P_4 = (2, 1, \cdot)$
\exists	$z := 1$	$P_5 = (2, 1, 1)$
	evaluation	\exists wins

Example

Formula

$$2 = 2$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
\exists	$x := 1$	$P_1 = (1, \cdot, \cdot)$
\forall	$y := 1$	$P_2 = (1, 1, \cdot)$
\exists	backtrack to $P_0 + x := 2$	$P_3 = (2, \cdot, \cdot)$
\forall	$y := 1$	$P_4 = (2, 1, \cdot)$
\exists	$z := 1$	$P_5 = (2, 1, 1)$
	evaluation	\exists wins

History

$$H := \bigcup_n P_n$$

G^0 : deductive system**Rules:**

- If there exists $(\vec{m}_h, \vec{n}_h) \in H$ such that $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{H \in \mathbb{W}_\Phi^0} \text{WIN}$$

- For all $i < h$, $(\vec{m}_i, \vec{n}_i) \in H$ and $m \in \mathbb{N}$:

$$\frac{H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n)\} \in \mathbb{W}_\Phi^0 \quad \forall n \in \mathbb{N}}{H \in \mathbb{W}_\Phi^0} \text{PLAY}$$

G¹: playing with realizability

Formulae structure

$$\begin{aligned}\Phi &\equiv \exists^N x_1 \forall^N y_1 \dots \exists^N x_h \forall^N y_h (f(\vec{x}_h, \vec{y}_h) = 0) \\ &\equiv \forall X_1 (\forall^N x_1 (\forall^N y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)\end{aligned}$$

G¹: playing with realizability

Formulæ structure

$$\Phi \equiv \exists^N x_1 \forall^N y_1 \dots \exists^N x_h \forall y_h (f(\vec{x}_h, \vec{y}_h) = 0)$$

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$$\Phi_{i-1} \equiv \forall X_i (\forall^N x_i (\forall^N y_i \Phi_i \Rightarrow X_i) \Rightarrow X_i)$$

$$\Phi_h \equiv \forall W (W(f(\vec{x}_h, \vec{y}_h)) \Rightarrow W(0))$$

G^1 : playing with realizability

Formulae structure

$$\Phi_0 \equiv \forall X_1 (\forall^N x_1 (\forall^N y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

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Realizability

$$\|A \Rightarrow B\| = \{u \cdot \pi : u \in |A| \wedge \pi \in \|B\|\}$$

$$\|\forall^N x A(x)\| = \bigcup_{n \in \mathbb{N}} \{\bar{n} \cdot \pi : \pi \in \|A(n)\|\}$$

G^1 : playing with realizability

Formulae structure

$$\Phi_0 \equiv \forall X_1 (\forall^N x_1 (\forall^N y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

Start :

- Eloise proposes t_0 to defend Φ_0
- Abelard proposes $u_0 \cdot \pi_0$ to attack Φ_0

move	p_i (\exists -position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$

G^1 : playing with realizability

Formulae structure

$$\Phi_0 \equiv \forall X_1 (\forall^N x_1 (\forall^N y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

move	p_i (\exists -position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$

Eloise reduces p_0 until

- $p_0 \succ u_0 \star \overline{m_1} \cdot t_1 \cdot \pi_0$
 - \rightsquigarrow she *can* decide to play (m_1, t_1) and ask for Abelard's answer
 - \rightsquigarrow Abelard *must* give $\overline{m_1} \cdot u' \cdot \pi'$.

G^1 : playing with realizability

Formulae structure

$$\Phi_0 \equiv \forall X_1 (\forall^N x_1 (\forall^N y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

move	p_i (\exists -position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$
1	$t_1 \star \bar{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$

Eloise reduces p_0 until

- $p_0 \succ u_0 \star \bar{m}_1 \cdot t_1 \cdot \pi_0$
 - ↪ she *can* decide to play (m_1, t_1) and ask for Abelard's answer
 - ↪ Abelard *must* give $\bar{n}_1 \cdot u' \cdot \pi'$.

G^1 : playing with realizability

Formulae structure

$$\Phi_{i-1} \equiv \forall X_i (\forall^N x_i (\forall^N y_i \Phi_i \Rightarrow X_i) \Rightarrow X_i)$$

move	p_i (\exists -position)	history
1	$t_1 \star \bar{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$
\vdots	\vdots	\vdots
i	$t_i \star \bar{n}_i \cdot u_i \cdot \pi_i$	$H_i := \{(m_i, n_i, u_i, \pi_i)\} \cup H_{i-1}$

Eloise reduces p_i until

- $p_i \succ u \star \bar{m} \cdot t \cdot \pi$ with $(\vec{m}_j, \vec{n}_j, u, \pi) \in H_j$ where $j < h$.
 - ↪ she *can* decide to play (m_{i+1}, t_{i+1})
 - ↪ Abelard *must* give $\bar{n}_i \cdot u' \cdot \pi'$.

G^1 : playing with realizability

Formulae structure

$$\Phi_h \equiv \forall W(W(f(\vec{x}_h, \vec{y}_h)) \Rightarrow W(0))$$

move	p_i (\exists -position)	history
1	$t_1 \star \bar{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$
\vdots	\vdots	\vdots
i	$t_i \star \bar{n}_i \cdot u_i \cdot \pi_i$	$H_i := \{(m_i, n_i, u_i, \pi_i)\} \cup H_{i-1}$

Eloise reduces p_i until

- $p_i \succ u \star \bar{m} \cdot t \cdot \pi$ with $(\vec{m}_j, \vec{n}_j, u, \pi) \in H_j$ where $j < h$.
 - ↪ she can decide to play (m_{i+1}, t_{i+1})
 - ↪ Abelard must give $\bar{n}_i \cdot u' \cdot \pi'$.
- $p_i \succ u \star \pi$ with $(\vec{m}_h, \vec{n}_h, u, \pi) \in H_j$
 - ↪ she wins iff $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$.

G^1 : formal definition

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{\langle p, H \rangle \in \mathbb{W}_\Phi^1} \text{WIN}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $p \succ u \star \vec{m} \cdot t \cdot \pi$:

$$\frac{\langle t \star \vec{n} \cdot u' \cdot \pi', H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\} \rangle \in \mathbb{W}_\Phi^1 \quad \forall(n', u', \pi')}{\langle p, H \rangle \in \mathbb{W}_\Phi^1} \text{PLAY}$$

Winning strategy

$t \in \Lambda_c$ s.t. for any handle $(u, \pi) \in \Lambda \times \Pi$:

$$\langle t \star u \cdot \pi, \{(\emptyset, \emptyset, u, \pi)\} \rangle \in \mathbb{W}_\Phi^1$$

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Specification result

Adequacy

If t is a winning strategy for G_Φ^1 , then $t \Vdash \Phi$

Proof (sketch):

- play a match with stacks from falsity value,
- conclude by anti-reduction.

Specification result

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Completeness of G^1

If the calculus is deterministic and substitutive, then if $t \Vdash \Phi$ then t is a winning strategy for the game G_Φ^1

Proof (sketch): by contradiction

- substitute Abelard's winning answers along the thread scheme,
- reach a winning position anyway.

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The general case

Loosing the substitution

quote

$$\text{quote} \star \varphi \cdot t \cdot \pi \succ t \star \bar{n}_\varphi \cdot \pi$$

- the calculus is no longer substitutive
- there are some wild realizers which are not winning strategies!

Consider $\Phi_{\leq} \equiv \exists^N x \forall^N y (x \leq y)$ and t_{\leq} s.t. :

$$t_{\leq} \star u \cdot \pi \succ T_0 \star \pi \succ u \star \bar{0} \cdot T_1 \cdot \pi$$

and :

$$T_1 \star \bar{n} \cdot u' \cdot \pi' \succ \begin{cases} \blacksquare \star \pi' & \text{if } u' \equiv T_0 \text{ and } \pi' \equiv \pi \\ u' \star \pi' & \text{otherwise} \end{cases}$$

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↷ Idea : I've already been there...

G^2 : non-substitutive case

↪ Idea : *I've already been there...*

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{\langle p, H \rangle \in \mathbb{W}_\Phi^1} \text{WIN}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $p \succ u \star \bar{m} \cdot t \cdot \pi$:

$$\frac{\langle t \star \bar{n} \cdot u' \cdot \pi', H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\} \rangle \in \mathbb{W}_\Phi^1 \quad \forall(n', u', \pi')}{\langle p, H \rangle \in \mathbb{W}_\Phi^1} \text{PLAY}$$

G^2 : non-substitutive case

↪ Idea : *I've already been there...*

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$, $\exists p \in \mathbf{P}$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$:

$$\overline{\langle \mathbf{P}, H \rangle} \in \mathbb{W}_{\Phi}^2 \text{ WIN}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $\exists p \in \mathbf{P}$, $p \succ u \star \bar{m} \cdot t \cdot \pi$:

$$\frac{\langle \{t \star \bar{n} \cdot u' \cdot \pi'\} \cup \mathbf{P}, H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\} \rangle \in \mathbb{W}_{\Phi}^2 \quad \forall (n', u', \pi')}{\langle \mathbf{P}, H \rangle \in \mathbb{W}_{\Phi}^2} \text{ P}$$

Specification result

Adequacy

If t is a winning strategy for G_{Φ}^2 , then $t \Vdash \Phi$

Proof (sketch):

- play a match with stacks from falsity value,
- conclude by anti-reduction.

Specification result

Adequacy

If t is a winning strategy for G_Φ^2 , then $t \Vdash \Phi$

Proof (sketch):

- play a match with stacks from falsity value,
- conclude by anti-reduction.

Completeness of G^2

If $t \Vdash \Phi$ then t is a winning strategy for the game G_Φ^2

Proof (sketch): by contradiction,

- build an increasing sequence $\langle P_i, H_i \rangle$ using \bigvee winning answers,
- define $\perp\!\!\!\perp := (\bigcup_{p \in P_\infty} \mathbf{th}(p))^c$,
- reach a contradiction.

Consequences

Proposition: Uniform winning strategy

There exists a term T such that if:

- $\mathcal{M} \vDash \exists x_1 \forall y_1 \dots f(\vec{x}, \vec{y}) = 0$
- θ_f computes f

then $T\theta_f$ is a winning strategy for $\exists x_1 \forall y_1 \dots f(\vec{x}, \vec{y}) = 0$.

Proof: Enumeration of \mathbb{N}^k , using θ_f to check whether we reached a winning position.

Theorem: Absoluteness

If Φ is an arithmetical formula, then

$$\exists t \in \Lambda_c, t \Vdash \Phi \quad \text{iff} \quad \mathcal{M} \vDash \Phi$$

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Comments & conclusion

About equality

$$\Phi_{2h} : \exists x_1 \forall y_1 \dots \exists x_h \forall y_h f(\vec{x}_h, \vec{y}_h) \neq 0$$

	$\ f(x) = 0\ $	$\ f(x) \neq 0\ $
$\mathcal{M} \models f(x) = 0$	$\ \forall X. X \rightarrow X\ $	\perp
$\mathcal{M} \models f(x) \neq 0$	$\Lambda_c \times \perp$	\emptyset

Uniform realizer

$\forall n \in \mathbb{N}$, there exists $t_n \in \Lambda_c$ s.t. $\forall f : \mathbb{N}^{2n} \rightarrow \mathbb{N}$,
 $\mathcal{M} \models \exists x_1 \forall y_1 \dots f(\vec{x}, \vec{y}) \neq 0 \Rightarrow t_n \Vdash \exists^N x_1 \forall^N y_1 \dots f(\vec{x}, \vec{y}) \neq 0$.

\Rightarrow t does not necessarily play according to the formula...

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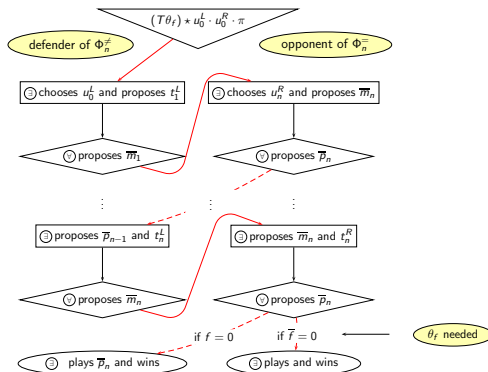
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\curvearrowright t does not necessarily play according to the formula...

Combining strategies

For all n , there exists a term T_n s.t. if θ_f computes f , then

$$T_n \theta_f \Vdash \Phi_n^{\neq} \Rightarrow \Phi_n^=$$


About absoluteness

- it was already known
- it extends to realizability algebras
- we now know even more :

Shoenfield barrier

Every Σ_2^1/Π_2^1 -relation is absolute for all *inner models* \mathcal{M} of ZF.

Krivine'14

There exists an ultrafilter on \mathbb{N}

Corollary

For any realizability algebra \mathcal{A} , $\mathcal{M}^{\mathcal{A}}$ contains a proper class \mathcal{M}' which is an *inner model* of ZF.

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Conclusion

What we did :

- We defined two games for substitutive and non-substitutive cases
- We proved equivalence between universal realizers and winning strategies
- It solved both specification and absoluteness problems

Further work :

- classes of formulæ compatible with games ?
- transformation $\mathbb{G}^1 \rightsquigarrow \mathbb{G}^2$ generic ?
+ some kind of *loss of innocence* ?
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Thank you for your attention.